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Conditional uniform time stable numerical solutions of coupled hyperbolic systems

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Abstract

This work deals with the uniform time stability of discrete numerical solutions of strongly coupled hyperbolic mixed problems. Using the Crank–Nicholson scheme and a Fourier discrete method the uniform time stability of the solution constructed is based on the spectral analysis of roots of the underlying algebraic matrix equation.

Keywords: Hyperbolic system; Uniform time stability; Numerical solution

1. Introduction and preliminaries

Coupled hyperbolic partial differential systems arise in many different fields; see [1] and references therein. In this work we consider mixed problems of the form

$$Au_{xx}(x,t) = u_{tt}(x,t), \quad 0 < x < 1, t > 0, \tag{1}$$
$$u(0,t) = 0, \quad t > 0, \tag{2}$$
$$Bu(1,t) + Cu_x(1,t) = 0, \quad t > 0, \tag{3}$$
$$u(x,0) = F(x), \quad 0 \le x \le 1, \tag{4}$$
$$u_t(x,0) = V(x), \quad 0 \le x \le 1, \tag{5}$$

where A, B, C are matrices in $\mathbb{C}^{s\times s}$ and the unknown $u(x,t)$ as well as functions $F(x)$ and $V(x)$ take values in $\mathbb{C}^s$. We assume that C is invertible and that there exists a real number μ such that

$$\mu \in \sigma(-C^{-1}B) \cap \mathbb{R}, \quad \mu < 1. \tag{6}$$

In [1], the authors proved that under the hypothesis

$$\alpha(A) = \min\{\mathrm{Re}(z); z \text{ eigenvalue of } A\} > 0, \tag{7}$$

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after considering the Crank–Nicholson difference scheme and the subdivision of the domain $[0, 1] \times [0, \infty[$ into equal rectangles of sides $h = \Delta x = 1/M$, $\Delta t = k$, a numerical solution of problem (1)–(6) is given by

$$u(mh, nk) = U(m, n) = \sum_{\ell=1}^{M-1} [(Z_0(\ell))^n P_\ell + (Z_1(\ell))^n Q_\ell] \sin(m\theta_\ell), \tag{8}$$

where

$$Z_0(\ell) = (I - \rho_\ell A/2)^{-1}[I + \rho_\ell A + i|\rho_\ell|^{1/2}\sqrt{A}\sqrt{I + \rho_\ell A/16}], \tag{9}$$

$$Z_1(\ell) = (I - \rho_\ell A/2)^{-1}[I + \rho_\ell A - i|\rho_\ell|^{1/2}\sqrt{A}\sqrt{I + \rho_\ell A/16}], \tag{10}$$

are solutions of the algebraic matrix equation

$$Z^2 - \left(I - \frac{\rho_\ell A}{2}\right)^{-1}\left(2I + \frac{\rho_\ell A}{2}\right) Z + \left(I - \frac{\rho_\ell A}{2}\right)^{-1} = 0, \tag{11}$$

$$\rho_\ell = -4r^2 \sin^2(\theta_\ell/2), \quad r = k/h,$$

where θ_ℓ is a root of the equation

$$\cot(M\theta) = \frac{\cos\theta - (1 - \mu/M)}{\sin\theta}, \tag{12}$$

in $J_\ell =]\frac{(\ell-1)\pi}{M}, \frac{\ell\pi}{M}[$, for $1 \le \ell \le M - 1$. Vectors P_ℓ and Q_ℓ are given by

$$P_\ell = (Z_0(\ell) - Z_1(\ell))^{-1} \frac{\sum_{s=1}^{M-1} [kv(s) - (Z_1(\ell) - I)f(s)] \sin(s\theta_\ell)}{\sum_{s=1}^{M-1} \sin^2(s\theta_\ell)}, \tag{13}$$

$$Q_\ell = (Z_0(\ell) - Z_1(\ell))^{-1} \frac{\sum_{s=1}^{M-1} [(Z_0(\ell) - I)f(s) - kv(s)] \sin(s\theta_\ell)}{\sum_{s=1}^{M-1} \sin^2(s\theta_\ell)}, \tag{14}$$

with

$$f(s) = F(sh), \quad v(s) = V(sh), \quad r = k/h, \quad h = 1/M. \tag{15}$$

The solution (8) is stable in the fixed station sense with respect to the time, i.e., given $T > 0$ and M positive integer, both fixed, $U(m, n)$ remains bounded as $n \to \infty$, $k > 0$, but with the restriction $1 \le n \le N$, $Nk = T$, for all $1 \le m \le M$. This means that the bound of $U(m, n)$ depends on the point T; see Section 5 of [1, p. 173–174].

In this work we show that if matrix A of (1) has real eigenvalues apart from (7), or equivalently

Every eigenvalue a of A is strictly positive, (16)

then the numerical solution (8) of problem (1)–(5) is uniformly stable with respect to the time, in the sense that given a fixed positive integer $M > 0$, with $h = 1/M$, one gets

$$\sup_{\substack{n \ge 1 \\ 1 \le m \le M}} \|U(m, n)\| \le L < +\infty, \tag{17}$$

where L is independent of m and n. If A is symmetric, then condition (7) is equivalent to (16), but condition (16) does not mean that matrix A is necessarily symmetric. For instance, a matrix A that is similar to a symmetric definite positive matrix B satisfies (16) but A may be non-symmetric; see [2, p. 106]. The concept of uniform stability with respect to the time is related to the concept of stability recently introduced in [3] for the study of coupled parabolic problems.

2. Uniform stability with respect to the time

Let us consider the numerical solution of problem (1)–(5) given by (8) under hypotheses (6) and (16). Let M be a positive integer with $h = \Delta x = 1/M$ fixed, and let

$$\beta(A) = \max\{a;\, a \text{ eigenvalue of } A\}. \tag{18}$$

In accordance with [1], matrices $Z_0(\ell)$ and $Z_1(\ell)$ are well defined for $|\rho_\ell| < \frac{16}{\beta(A)}$, or

$$k\, M = r < \frac{2}{\sin(\theta_\ell/2)\sqrt{\beta(A)}}. \tag{19}$$

Note that condition (19) holds true if

$$\Delta t = k < \frac{2}{M\, \sqrt{\beta(A)}}. \tag{20}$$

Now, by the spectral mapping theorem [4, p. 569], the set of eigenvalues of $Z_0(\ell)$ is given by

$$\begin{aligned}\sigma\,(Z_0(\ell)) &= \left\{\frac{1 + a\ \rho_\ell/4 + i|\rho_\ell|^{1/2}\sqrt{a}\sqrt{1 + a\ \rho_\ell/16}}{1 - a\ \rho_\ell/2}\ ;\ a \in \sigma(A)\right\}\\ &= \left\{\frac{4 + a\ \rho_\ell}{2\,(2 - a\ \rho_\ell)} + \frac{2\,i|\rho_\ell|^{1/2}\sqrt{a}\sqrt{1 + a\ \rho_\ell/16}}{2 - a\ \rho_\ell}\ ;\ a \in \sigma(A)\right\}.\end{aligned}$$

As $|\rho_\ell| = -\rho_\ell = 4r^2 \sin^2(\theta_\ell/2)$, note that

$$\begin{aligned}\left|\frac{4 + a\ \rho_\ell}{2\,(2 - a\ \rho_\ell)} + \frac{2\,i|\rho_\ell|^{1/2}\sqrt{a}\sqrt{1 + a\ \rho_\ell/16}}{2 - a\ \rho_\ell}\right|^2 &= \frac{(4 + a\ \rho_\ell)^2 + |\rho_\ell| a\,(16 + a\ \rho_\ell)}{4\,(2 - a\ \rho_\ell)^2}\\ &= \frac{16 + 8\,a\ \rho_\ell + a^2\ \rho_\ell^2 - a\ \rho_\ell\,(16 + a\ \rho_\ell)}{4\,(2 - a\ \rho_\ell)^2}\\ &= \frac{2}{2 - a\ \rho_\ell} = \frac{2}{2 + 4\,r^2\,a \sin^2(\theta_\ell/2)}\\ &= \frac{1}{1 + 2\,r^2\,a \sin^2(\theta_\ell/2)}\\ &\le \frac{1}{1 + 2\,r^2\,\alpha(A) \sin^2(\theta_\ell/2)}\\ &= \frac{1}{1 + 2\,k^2 M^2\,\alpha(A) \sin^2(\theta_\ell/2)}.\end{aligned} \tag{21}$$

Hence the spectral radius $\rho(Z_0(\ell))$ of $Z_0(\ell)$ satisfies

$$\rho\,(Z_0(\ell)) \le \sqrt{\frac{1}{1 + 2k^2 M^2 \alpha(A) \sin^2(\theta_\ell/2)}}\ <\ 1.$$

In an analogous way one gets

$$\rho\,(Z_1(\ell)) \le \sqrt{\frac{1}{1 + 2\,k^2 M^2\,\alpha(A) \sin^2(\theta_\ell/2)}}\ <\ 1.$$

By Theorem 1.3.6 of [5, p. 23], there exists a norm $\|\cdot\|$ in $\mathbb{C}^{s\times s}$ such that $\|Z_0(\ell)\| < 1$ and $\|Z_1(\ell)\| < 1$, for $1 \le \ell \le M - 1$. On the other hand, by [1, p. 174] one gets

$$\|P_\ell\| = \mathrm{O}(1), \quad \|Q_\ell\| = \mathrm{O}(1), \quad \text{as } k \to 0,\ 1 \le \ell \le M - 1.$$

Hence, $U(m, n)$ defined by (8) satisfies

$$\sup_{\substack{n\geq 1\\ 1\leq m\leq M}} \|U(m,n)\| \leq \sum_{\ell=1}^{M-1} \left(\|P_\ell\| + \|Q_\ell\| \right) \leq M\ \mathrm{O}(1) < +\infty,$$

and thus $U(m, n)$ is uniformly stable with respect to the time, under the hypothesis (20). Thus the following result has been established.

Theorem 2.1. *Assume that matrix A of (1) satisfies condition (16). Then the numerical solution $U(m, n)$ of problem (1)–(6) given by (8) is conditionally uniformly stable with respect to the time. Fixing a positive integer M and $h = \Delta x = 1/M$, the condition for $\Delta t = k$, in order to guarantee the uniform time stability, is given by (20).*

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